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# DIFFERENTIAL EQUATIONS AND IMPLICIT FUNCTIONS IN INFINITELY MANY VARIABLES

By William L. Hart
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO

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The paper of which this note gives a brief abstract, has three main divisions. In the first section certain fundamental theorems are developed concerning a type of real-valued functions of infinitely many real variables. In the second section there is considered the problem of infinite systems of ordinary differential equations.

$$\frac{dx_i}{dt} = f_i\,(x_1, x_2, \; . \; . \; . \; ; t), \qquad x_i\,(t_0) = a_i, \qquad i = 1, 2, \; . \; . \; ., \qquad (1)$$

in which the $f_i$ are of the type treated in the initial theorems. In the third section of the paper the fundamental problem of implicit function theory in this field is discussed for a system of equations

$$\begin{aligned} &f_i\,(x_1, x_2, \; . \; . \; . \; ; y_1, y_2, \; . \; . \; .) = 0, \qquad i = 1, 2, \; . \; . \; ., \\ &f_i\,(a_1, a_2, \; . \; . \; . \; ; b_1, b_2, \; . \; . \; .) = 0, \end{aligned} \qquad (2)$$

where the $x_j$ are independent variables and the $y_j$ are to be determined. The results of all three sections of the paper include as special cases the corresponding theorems on functions of a finite number of variables.

The region of real points $\xi = (x_1, x_2, \; . \; . \; .)$ in the space of infinitely many dimensions in which the functions considered are supposed defined, is of the type specified by

$$R : |\, x_i - a_i \,| \leqq r_i, \qquad i = 1, 2, \; . \; . \; ., \qquad 0 < r \leqq r, \qquad r \text{ finite.}$$

In the first part of the investigation theorems are derived for functions *completely continuous* according to the

Definition 1. *A function* $f(\xi)$ *defined in* $R$ *is completely continuous at a point* $\xi^{(0)}$*—in notation,* $C_1$ $(\xi^{(0)})$*—if whenever*

$$\lim_{n=\infty} x_{in} = x_i^{(0)}, \qquad i = 1, 2, \ldots, \tag{3}$$

*it follows that*

$$\lim_{n=\infty} f(\xi_n) = f(\xi^{(0)}), \qquad \xi_n = (x_{1n}, x_{2n}, \ldots).$$

This concept of complete continuity is the same as that of '*Vollstetigkeit*' which has been much used by Hilbert and his followers. However, in the present paper the functions satisfying Definition (1) are supposed defined for a region of points different from that used by Hilbert and entirely different applications of the concept are made.

A function $f(\xi)$ is said to be $C_0$ at $\xi^{(0)}$ if it has the weaker continuity resulting from Definition (1) when in (3) it is assumed that the convergence is uniform with respect to $i$.

In deriving theorems on completely continuous functions important use is made of the following condensation lemma:[1]

Lemma 1. *Let* $S = (\xi_n : n = 1, 2, \ldots)$ *be a sequence of points of* $R$. *Then there exists a point* $\xi'$ *of* $R$ *and an infinite sub-sequence* $\xi'_n$ $(n = 1, 2, \ldots)$ *of* $S$ *such that* $\lim_{n=\infty} x'_{in} = x'_i$ $(i = 1, 2, \ldots)$.

The theorems obtained, of which the more important ones are given below, are derived by methods similar to those used in obtaining corresponding results in classical analysis.

Theorem 1. *To state that* $f(\xi)$ *is* $C_0$ *at* $\xi^{(0)}$ *is equivalent to saying that, for every* $e > 0$ *there exists a number* $d_e > 0$ *such that for every* $\xi'$ *of* $R$ *satisfying* $| x'_i - x_i^{(0)} | \leqq d_e$, $i = 1, 2, \ldots$, *there is the inequality* $f(\xi') - f(\xi^{(0)}) | \leqq e$.

On taking this equivalent definition of the $C_0$ property, there is obtained

Theorem 2. *If* $f(\xi)$ *is* $C_1$ *at every point of* $R$, *then* $f(\xi)$ *possesses the* $C_0$ *property uniformly in* $R$.

The analogues of two fundamental theorems from the theory of continuous functions are found in the next two theorems.

Theorem 3. *If* $f_n(\xi)$ $(n = 1, 2, \ldots)$ *are* $C_1(\xi^{(0)})$ *while also* $\lim_{n=\infty} f_n(\xi) = f(\xi)$ *uniformly for* $\xi$ *in* $R$, *then* $f$ *is* $C_1$ $(\xi^{(0)})$.

Theorem 4. *If* $f(\xi)$ *is* $C_1$ *at all points of* $R$, *then* $| f(\xi) |$ *is finitely limited and* $f$ *attains its upper and lower bounds in* $R$.

As a foundation for much of the work in the sections on differential equations and implicit functions, there is established for infinitely many variables an analogue of Taylor's Theorem with $n$ terms and an integral form for the remainder. The 'mean value' case of the general theorem is

Theorem 5. *Suppose that the function* $f(\xi)$ *and its partial derivatives* $\partial f/\partial x_i$ $(i = 1, 2, \ldots)$ *are* $C_1$ *in* $R$ *and that* $\Sigma_{j=1}^{\infty} r_j \,|\, \partial f(\xi)/\partial x_j \,|$ *converges uniformly for all* $\xi$ *in* $R$. *Then for* $(\xi, \xi')$ *of* $R$,

$$f(\xi') - f(\xi) = \sum_{j=1}^{\infty} (x'_j - x_j) \int_0^1 \frac{\partial f[\xi + u(\xi' - \xi)]}{\partial x_j} du.$$

Systems of differential equations of the form (1) were first considered by H. Von Koch[2] who treated a case of an analytic type. A very similar problem was considered later by F. R. Moulton.[3] Infinite systems of a linear form were discussed by E. H. Moore[4] as a special case of a more general investigation made in the sense of Moore's General Analysis.[5]

In the present paper, the problem (1) is treated by a generalized *Picard Approximation* method. The coördinates $x_{ik}(t)$ of the approximations $\xi_k(t)$ to a solution of (1) are defined formally by the equations

$$\begin{aligned} \xi_0(t) &= (a_1, a_2, \ldots), \\ x_{ik}(t) &= x^{t}{}_{0} + \int_0^t f_i[\xi_{k-1}(t), t]\, dt, \qquad k = 1, 2, \ldots. \end{aligned} \tag{4}$$

The existence theorem obtained is

Theorem 6. *Suppose in* (1) *that the* $f_i$ *are defined and* $C_1$ *in*

$$T: |\, t - t_0 \,| \leqq r_0;\ R: |\, x_i - a_i \,| \leqq r_i \quad (0 < r_i < r;\ r \text{ finite}), \tag{5}$$

*and that there exist positive functions* $A_{ij}(t, \xi, \xi')$ *defined and* $C_1$ *for* $t$ *in* $T$ *and* $(\xi, \xi')$ *in* $R$. *Assume that*

$$|\, f_i(\xi, t) - f_i(\xi', t) \,| \leqq \sum_{j=1}^{\infty} A_{ij}(t, \xi, \xi') \,|\, x'_j - x_j \,|. \tag{6}$$

*Suppose, moreover, that for* $(i = 1, 2, \ldots)$ *and for all admissible values of* $(t, \xi, \xi')$

$$\sum_{j=1}^{\infty} r_j A_{ij}(t, \xi, \xi') = V_i(t, \xi, \xi')$$

*converges uniformly, and that the maxima* $M_i$ *of the* $|f_i(\xi, t)|$ *and the maxima* $K_i$ *of the* $V_i$ *satisfy* $M_i \leqq r_i M$, $2K_i \leqq r_i K$, *with* $K$ *and* $M$ *finite. Then the approximations* (4) *exist and converge to a function* $\xi(t)$ *for* $|\, t - t_0 \,|$ *sufficiently small. Moreover,* $x_i(t)$ *is continuous in* $t$ *and* $\xi(t)$ *is the unique continuous solution of* (1).

The linear system of Moore is not related to the system (1) of the preceding theorem but the results of Moulton, when restricted to reals, are a special case of the conclusion of Theorem 6. In addition to the results of Moulton, however, it follows that the *unique analytic solution* he obtained is, for real values of $t$, the *only continuous solution of* the system he treated.

Under certain hypotheses in addition to those of Theorem 6, it is proved that the solution $\xi\,(t)$ has continuation properties which reduce, for the finite case, to the ordinary theorem on the existence of a solution extending to the boundary of the given region of definition of the system.

It is also shown, by the aid of Theorem 5, that the formal hypothesis (6) can be replaced by an assumption concerning the existence and the values of the partial derivatives $\partial f_i/\partial x_j$ $(i, j = 1, 2, \ldots)$.

Infinite systems of equations of the form (2) have been considered by H. Von Koch[6] and R. d'Adhemar.[7] Von Koch treated a system of an analytic type defined in the field of complex numbers and established the existence of an analytic solution. His work, however, is valid only if the sum of the numbers $r_i$ of the region similar to $R$ in which his system is defined converges in a very special manner. R. d'Adhemar treated a special type which arose in a problem he considered in integral equation theory. The results of Von Koch, when restricted to reals, are a special case of the theorem stated below.

In the solution of the system (2), considered in the present paper, infinite sets of linear equations enter in a fundamental fashion. Such hypotheses are imposed that these linear systems come under the theory of infinite systems of linear equations with normal determinants. The method of solution of (2) for the $y_j$ is related to that used by Goursat[9] in his solution of the finite case by a method of successive approximations. An analogue for (2) of the fundamental theorem on implicit functions in the finite case, is obtained in the following form:

Theorem 7. *Suppose that, in* (2), *the functions* $f_i$ *and* $\partial f_i/\partial y_j$ $(i, j = 1, 2, \ldots)$ *are defined and* $C_1$ *for all points in*

$$S: | x_i - a_i | \leqq r_i;\ T: | y_i - b_i | \leqq r_i,\ i = 1, 2, \ldots, 0 < r_i \leqq r, \quad (7)$$

*and that the maxima* $M_i$ *of the* $| f_i(\xi, \eta) |$ $(\eta = y_1, y_2, \ldots)$ *satisfy* $M_i \leqq r_i M$ *(*$M$ *finite). Assume that for all* $(\xi, \eta)$ *in* (7)

$$\sum_{i,j=1}^{\infty} \left| d_{ij} - \frac{\partial f_i(\xi, \eta)}{\partial y_j} \right|, \qquad d_{ij} = 0,\ i \neq j;\ d_{ii} = 1, \quad (8)$$

*converges uniformly. Suppose that the normal infinite determinant*

$$\Delta = \left| \frac{\partial f_i\,(\alpha, \beta)}{\partial y_j} \right|_{i,j=1,2,\ldots} \neq 0, \qquad \alpha = (a_1, a_2, \ldots), \beta = (b_1, b_2; \ldots). \quad (9)$$

*Suppose that there exists a number B such that*

$$\sum_{k=1}^{\infty} \left| D_{ki} \right| \leqq Br_i \qquad i = 1, 2, \ldots ,$$

*where* $D_{ki}$ *is the cofactor of the element* $\partial f_k/\partial y_i$ *in* (9). *Let it be assumed that for every* $e > 0$ *there can be found a number* $d_e > 0$ *such that, for* $| x_j - a_j | \leqq d_e$ $(j = 1, 2, \ldots)$ *it follows that* $| f_i(\xi, \beta) | \leqq e$ $(i = 1, 2, \ldots)$. *Then there exist positive constants* $(c, d)$, $0 < d \leqq 1$, $0 < c \leqq d$, *such that to every* $\xi$ *satisfying* $| x_j - a_j | \leqq cr_j$ *there corresponds one and only one solution of* (2) *in the region defined by* $| y_j - b_j | \leqq dr_j$. *Furthermore, the solution* $\eta\,(x_1, x_2, \ldots)$ *so determined is* $C_1$ *for all* $\xi$ *satisfying* $| x_j - a_j | \leqq cr_j$.

The problem of obtaining continuation properties for the solution $\eta\,(\xi)$ is not developed in the present paper because the form of the hypotheses used here does not readily lend itself to this generalization.

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[1] Cf. an equivalent theorem proved in Bolza's *Vorlesungen über Variationsrechnung*, p. 423, part (b).

[2] *Öfversigt af Kongliga Vetenskaps Akademiens Förhandlingar*, **56**, 395–411 (1899).

[3] *These* PROCEEDINGS, **1**, 350 (1915).

[4] *Atti dei IV Congresso Internazionale dei Mathematici*, **2**, 98 (Roma, 6–11 Aprile, 1908).

[5] Introduction to a Form of General Analysis, *New Haven Mathematical Colloquium*.

[6] *Paris, Bull. Soc. Math.*, **27**, 215 (1899).

[7] *Ibid.*, **36**, 95 (1908).

[8] Cf. Kowalewski, *Einfürung in die Determinanten-Theorie*.

[9] *Paris, Bull. Soc. Math.*, **31**, 184 (1903).

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# THE SEX OF PARTHENOGENETIC FROGS

By Jacques Loeb

ROCKEFELLER INSTITUTE FOR MEDICAL RESEARCH, NEW YORK

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When it was established that larvae could be produced from unfertilized eggs by chemical methods,[1] a number of secondary problems arose. Foremost among these was the question whether or not the organisms thus produced were capable of developing into normal adults. This was at first considered improbable, especially by those who accepted Oscar Hertwig's statement that fertilization consisted in the fusion of the egg and sperm nucleus. Since no such fusion takes place in artificial parthenogenesis, it became obvious that either Hertwig's definition of fertilization was wrong or that artificial parthenogenesis was merely a pathological phenomenon not capable of leading to the formation of a normal organism.